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# Galilean Invariance and Magnetic Charge

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*Abstract*

The Galilean and 'dual' invariant electrodynamics with magnetic charges is formulated. The definition of the main feature of relativistic electromagnetism is given. Consideration of different aspects of Galilean electromagnetism with magnetic charges is presented. It is shown in particular that the conclusion of Bacry & Kubar-Andre (1973) that the existence of the magnetic monopole is incompatible with Galilean invariance in general appears to be incorrect.

## 1.

The Maxwell equations and electromagnetic force for the system of dual charged particles (particles endowed with both electric ($q$) and magnetic ($g$) charges) in MKSA units are:

$$\begin{aligned} \nabla \times \mathbf{H} &= \epsilon_0 \frac{\partial E}{\partial t} + \mathbf{j}_q, \qquad & \nabla . \mathbf{E} &= \rho_q/\epsilon_0 \\ \nabla \times \mathbf{E} &= -\mu_0 \frac{\partial H}{\partial t} - \mathbf{j}_g, \qquad & \nabla . \mathbf{H} &= \rho_g/\mu_0 \end{aligned} \tag{1.1}$$

$$\mathbf{F} = \int d^3r \{\rho_q \mathbf{E}(\mathbf{r}) + \rho_g \mathbf{H}(\mathbf{r}) + \mu_0 \mathbf{j}_q \times \mathbf{H}(\mathbf{r}) - \epsilon_0 \mathbf{j}_g \times \mathbf{E}(\mathbf{r})\} \tag{1.2}$$

There $\mathbf{j}_q(\mathbf{j}_g)$ is the density of electric (magnetic) current, $\rho_q(\rho_g)$ is the density of electric (magnetic) charge, $(c\rho, \mathbf{j})$ is a current four-vector, $\mathbf{j}_q = \Sigma_i \mathbf{j}_{qi}$, $\mathbf{j}_g = \Sigma_i \mathbf{j}_{gi}$, $\rho_q = \Sigma_i \rho_{qi}$, $\rho_g = \Sigma_i \rho_{g_i}$ and subscript ($i$) indicates a dual charged particles with charges $q_i$ and $g_i$, and $\epsilon_0 \mu_0$ (two constants) are defined in the usual way.

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The Maxwell equations (1.1) can be solved by means of two independent potentials $(c\varphi, \mathbf{A})$ and $(c\phi, \mathbf{B})$.

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t} - \nabla \times \mathbf{B} = \mathbf{E}_q + \mathbf{E}_g$$

$$\mu_0 \mathbf{H} = \nabla \times \mathbf{A} - \nabla\phi - \frac{\partial \mathbf{B}}{\partial t} = (\mathbf{H}_q + \mathbf{H}_g)\mu_0 \tag{1.3}$$

where

$$\mathbf{E}_q = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{E}_g = -\nabla \times \mathbf{B}, \qquad \mu_0 \mathbf{H}_q = \nabla \times \mathbf{A},$$

$$\mu_0 \mathbf{H}_g = -\nabla\phi - \frac{\partial \mathbf{B}}{\partial t}$$

and subscripts '$q$', '$g$' of the fields $\mathbf{E}$, $\mathbf{H}$ indicate the type of source that produce these fields.

The system of equations (1.1) falls, in this case, into two systems of equations:

$$\begin{aligned} \nabla \times \mathbf{H}_q &= \epsilon_0 \frac{\partial \mathbf{E}_q}{\partial t} + \mathbf{j}_q, &\qquad \nabla . \mathbf{E}_q &= \rho_q/\epsilon_0 \\ \nabla \times \mathbf{E}_q &= -\mu_0 \frac{\partial \mathbf{H}_q}{\partial t}, &\qquad \nabla . \mathbf{H}_q &= 0 \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} \nabla \times \mathbf{H}_g &= \epsilon_0 \frac{\partial \mathbf{E}_g}{\partial t}, &\qquad \nabla . \mathbf{E}_g &= 0 \\ \nabla \times \mathbf{E}_g &= -\mu_0 \frac{\partial \mathbf{H}_g}{\partial t} - \mathbf{j}_g, &\qquad \nabla . \mathbf{H}_g &= \rho_g/\mu_0 \end{aligned} \tag{1.5}$$

As has been shown (Le Bellac & Levy-Leblond, 1973; Penfield & Haus, 1967) there exist two different Galilean limits of electrodynamics: the 'electric' and 'magnetic' limit. These limits are based on the following conditions: (a) $c|\rho| \gg |\mathbf{j}|$ so $|\mathbf{E}| \gg c\mu_0|\mathbf{H}|$; (b) $c|\rho| \ll |\mathbf{j}|$ so $|\mathbf{E}| \ll c\mu_0|\mathbf{H}|$, respectively.

For the dual charged particles 'electric' limit corresponds to realisation of the following conditions:

$$\begin{aligned} c|{}^e\rho_q| &\gg |{}^e\mathbf{j}_q|, &\qquad |{}^e\mathbf{E}_q| &\gg c\mu_0|{}^e\mathbf{H}_q| \\ c|{}^e\rho_g| &\gg |{}^e\mathbf{j}_g|, &\qquad |{}^e\mathbf{E}_g| &\ll c\mu_0|{}^e\mathbf{H}_g| \end{aligned} \tag{1.6}$$

The Maxwell equations and Lorentz force have the following form in this limit:

$$\nabla \times {}^e\mathbf{H} = \epsilon_0 \frac{\partial^e \mathbf{E}_q}{\partial t} + {}^e\mathbf{j}_q, \qquad \nabla . {}^e\mathbf{E} = {}^e\rho_q/\epsilon_0$$
$$\nabla \times {}^e\mathbf{E} = -\mu_0 \frac{\partial^e \mathbf{H}_g}{\partial t} - {}^e\mathbf{j}_g, \qquad \nabla . {}^e\mathbf{H} = {}^e\rho_g/\mu_0 \tag{1.7}$$

$${}^e\mathbf{F} = \int d^3r\, \{{}^e\rho_q {}^e\mathbf{E}(\mathbf{r}) + {}^e\rho_g {}^e\mathbf{H}(\mathbf{r}) + \mu_0 {}^e\mathbf{j}_q \times {}^e\mathbf{H}_g(\mathbf{r}) - \epsilon_0 {}^e\mathbf{j}_g \times {}^e\mathbf{E}_q(\mathbf{r})\} \tag{1.8}$$

and they are invariants under the following Galilean transformations of fields and sources:

$${}^e\mathbf{E}' = {}^e\mathbf{E} + \mu_0 \mathbf{v} \times {}^e\mathbf{H}_g, \qquad {}^e\mathbf{H}' = {}^e\mathbf{H} - \epsilon_0 \mathbf{v} \times {}^e\mathbf{E}_q$$
$${}^e\rho'_{q(g)} = {}^e\rho_{q(g)}, \qquad {}^e\mathbf{j}'_{q(g)} = \mathbf{j}_{q(g)} - \mathbf{v} . {}^e\rho_{q(g)} \tag{1.9}$$

and Galilean transformation of the spatio-temporal gradient:

$$\frac{1}{c}\frac{\partial}{\partial t'} = \frac{1}{c}\frac{\partial}{\partial t} + \frac{1}{c}\mathbf{v} . \nabla \tag{1.10}$$
$$\nabla' = \nabla$$

Superscript $e$ indicates the 'electric' limit to which correspond equations (1.7) and (1.8) and transformations (1.9). The field vectors ${}^e\mathbf{E}$, ${}^e\mathbf{H}$ in (1.7)-(1.9) are a superposition of the fields ${}^e\mathbf{E}_q$, ${}^e\mathbf{E}_g$ and ${}^e\mathbf{H}_q$, ${}^e\mathbf{H}_g$. The transformations (1.9) can be derived from the usual Lorentz transformations of fields and sources taking into account condition (1.6) and the limit $c \to \infty$. First we get the Galilean limit of equations (1.4) and (1.5) and then make the transition to system (1.7).

In the 'magnetic' limit (superscript '$m$') we have:

$$c|{}^m\rho_q| \ll |{}^m\mathbf{j}_q|, \qquad |{}^m\mathbf{E}_q| \ll c\mu_0 |{}^m\mathbf{H}_q|$$
$$c|{}^m\rho_g| \ll |{}^m\mathbf{j}_g|, \qquad |{}^m\mathbf{E}_g| \gg c\mu_0 |{}^m\mathbf{H}_g| \tag{1.11}$$

In this limit Galilean transformations of fields and sources have the following form:

$${}^m\mathbf{E}' = {}^m\mathbf{E} + \mu_0 \mathbf{v} \times {}^m\mathbf{H}_q, \qquad {}^m\mathbf{H}' = {}^m\mathbf{H} - \epsilon_0 \mathbf{v} \times {}^m\mathbf{E}_g$$
$${}^m\rho'_{q(g)} = {}^m\rho_{q(g)} - \epsilon_0\mu_0 \mathbf{v} . {}^m\mathbf{j}_{q(g)}, \; {}^m\mathbf{j}'_{q(g)} = {}^m\mathbf{j}_{q(g)} \tag{1.12}$$

The Maxwell equations and the Lorentz force in this limit have the form:

$$\nabla \times {}^m\mathbf{H} = \epsilon_0 \frac{\partial^m \mathbf{E}_g}{\partial t} + {}^m\mathbf{j}_q, \qquad \nabla . {}^m\mathbf{E} = {}^m\rho_q/\epsilon_0$$
$$\nabla \times {}^m\mathbf{E} = -\mu_0 \frac{\partial^m \mathbf{H}_q}{\partial t} - {}^m\mathbf{j}_g, \qquad \nabla . {}^m\mathbf{H} = {}^m\rho_g/\mu_0 \tag{1.13}$$

$$ {}^{m}\mathbf{F} = \int d^3r\, \{{}^{m}\mathbf{p}_q \cdot {}^{m}\mathbf{E}_g(\mathbf{r}) + {}^{m}\rho_g \cdot {}^{m}\mathbf{H}_q(\mathbf{r}) + \mu_0 {}^{m}\mathbf{j}_q \times {}^{m}\mathbf{H}(\mathbf{r}) - \epsilon_0 {}^{m}\mathbf{j}_g \times {}^{m}\mathbf{E}(\mathbf{r})\} \tag{1.14} $$

While proving the Galilean invariance of (1.14) we made use of (1.13) and of the fact that the surface integral of the term ${}^{m}\mathbf{E}_q \times {}^{m}\mathbf{H}_g$ (or ${}^{m}\mathbf{E}_g \times {}^{m}\mathbf{H}_q$) is vanishing. In the 'magnetic' limit, as can be seen from (1.13), ${}^{m}\rho_{q(g)}$ and ${}^{m}\mathbf{j}_{q(g)}$ do not obey the continuity equation in contrast to the 'electric' limit.

If we put $\mathbf{j}_g = 0$, $\rho_g = 0$ and exclude from (1.7) to (1.9) and (1.12) to (1.14) the field vectors with subscript '$g$' and omit subscript '$q$' of the fields $\mathbf{E}, \mathbf{H}$ we get 'electric' and 'magnetic' limits of electrodynamics in Galilean invariant form with the presence of only electric sources.

The Maxwell equations and the Lorentz force in both limits are invariants under the following 'dual' transformations.

$$ \begin{aligned} \epsilon_0 {}^{d}\mathbf{E}_q &\to \epsilon_0 {}^{d}\mathbf{E}_q \cos\theta + \mu_0 {}^{d}\mathbf{H}_g \sin\theta, & \mu_0 {}^{d}\mathbf{H}_q &\to \mu_0 {}^{d}\mathbf{H}_q \cos\theta - \epsilon_0 {}^{d}\mathbf{E}_g \sin\theta \\ \epsilon_0 {}^{d}\mathbf{E}_g &\to \epsilon_0 {}^{d}\mathbf{E}_g \cos\theta + \mu_0 {}^{d}\mathbf{H}_q \sin\theta, & \mu_0 {}^{d}\mathbf{H}_g &\to \mu_0 {}^{d}\mathbf{H}_g \cos\theta - \epsilon_0 {}^{d}\mathbf{E}_q \sin\theta \end{aligned} \tag{1.15} $$

and corresponding transformations of the sources $\rho_q, \rho_g$ and $\mathbf{j}_q, \mathbf{j}_g$. Superscript '$d$' in (1.15) indicates the 'electric' '$e$' or 'magnetic' '$m$' limits.

We do not intend to discuss here the physical peculiarities of both Galilean formulations† of electrodynamics. The reason is that for the case of electric sources alone it was perfectly done by Le Bellac & Levy-Leblond (1973). Taking into consideration the 'dual' symmetry the discussion for the magnetic sources immediately follows. We only note that the main feature of Galilean formulations is that the force between the current and moving charge in the 'electric' limit and the static force between the charges in the 'magnetic' limit is absent.

## 2.

During the investigation of Galilean limits of electrodynamics with electric sources alone it turned out, rather unexpectedly, that we cannot give the precise meaning of what we refer to in relativistic aspects of electromagnetism (Le Bellac & Levy-Leblond, 1973). But the situation is different, as will be seen, if we take into account magnetic sources. In fact as can be seen from the field equations and equations of motion in both Galilean limits of electrodynamics, the electric and magnetic fields produced by electric and by magnetic sources have a different physical nature.‡ The introduction in Galilean electromagnetism (in both limits) of the following principle—it is impossible to determine experimentally the difference between electric and magnetic fields of

† Considering the definition (1.3) we can define the fields $\mathbf{E}, \mathbf{H}$ in both limits through the potentials $\mathbf{A}, \mathbf{B}, \phi, \varphi$ and discuss the Lagrange formulation of the theory on this basis.

‡ For example, in the 'electric' limit there exists a magnetic field produced by electric sources. But this field, contrary to the magnetic field of magnetic sources, has no effect on the electric charge in motion.

electric sources and the corresponding fields of magnetic sources—indicates the necessity of transition to the electrodynamics of Maxwell-Lorentz, where the fields $\mathbf{E}_q$, $\mathbf{E}_g$ and $\mathbf{H}_q$, $\mathbf{H}_g$ are equivalent in their physical appearance. But the above-mentioned principle has never been formulated before in a manifest form in electrodynamics, though it essentially reflects the relativistic aspects of Maxwell electromagnetism.

From the formal point of view, the correctness of this principle demands that the equation of motion and field equations must be invariant under the following substitutions (not necessarily simultaneous):

$$\mathbf{E}_q \leftrightarrow \mathbf{E}_g, \qquad \mathbf{H}_q \leftrightarrow \mathbf{H}_g$$

It can be satisfied only in the electrodynamics of Maxwell-Lorentz, and the transition of Galilean electromagnetism into Maxwell electromagnetism can be achieved by omitting the subscripts '$q$', '$g$' of the field vectors. If one omits the subscripts '$q$' and '$g$' of the fields $\mathbf{E}_{q(g)}$, $\mathbf{H}_{q(g)}$ (i.e. no distinction made about these fields), we pass from the Galilean transformations of fields (1.9) and (1.12) to the following transformations:

$$\begin{aligned} \mathbf{E}' &= \mathbf{E} + \mu_0 \mathbf{v} \times \mathbf{H} \\ \mathbf{H}' &= \mathbf{H} - \epsilon_0 \mathbf{v} \times \mathbf{E} \end{aligned} \tag{2.1}$$

These transformations are usually used in the quasi-relativistic consideration of electromagnetic phenomena. As was indicated by Le Bellac & Levy-Leblond (1973), and as can be seen from our derivation of the above-mentioned transformations, they do not correspond to the correct Galilean transformation of the fields. On the basis of the above-mentioned it was stated (Le Bellac & Levy-Leblond (1973)) that these transformations have no precise physical meaning and their use should be avoided. But from our investigation the following possibility of interpretation of these transformations evolves the consideration of (2.1) with the Maxwell equations (1.3) is a relativistic modification (e.g. taking into account the effect in the first order by $v/c$) of Galilean electromagnetism.

If we omit subscripts '$q$', '$g$' of field vectors the transformations (1.15) are the usual dual transformations in electrodynamics. Dual transformations of the field vectors $\mathbf{E}$, $\mathbf{H}$ can be induced by two types of dual transformations for their components $\mathbf{E}_q$, $\mathbf{E}_g$ and $\mathbf{H}_q$, $\mathbf{H}_g$. There can be transformations between electric and magnetic fields of one type (e.g. only with subscripts '$q$' or '$g$') or transformations of the type (1.15). Only the first type of transformations expresses the symmetry between electric and magnetic fields that are inherent in relativistic electromagnetism. The latter type of transformation reflects, as one can see, the symmetry of Galilean electromagnetism.

Supposing that electrodynamics is first formulated as Galilean invariant theory, the formulated principle above might and does serve as the basic for the transition to the relativistic (Maxwell-Lorentz) consideration of electromagnetism. This approach can be realised without the introduction of a new particle—the Dirac monopole.† As has been stated, the Maxwell-Lorentz electrodynamics can be formulated in dual symmetrical form whilst considering charged particles as dual charged particles under the condition of universality of ratio of their charges $g/q$ (see, for example, Schwinger, 1966; Strazhev, 1972). In this case one can remember an ingenious paper of Hertz (1884). Hertz, on the basis of introducing the principle of unity of all physical forces in electromagnetic phenomena and simultaneous introduction of magnetic sources, passed from the equations of electrodynamics of Weber-Neuman (theory of action at distance) to the Maxwell equations. The analysis of this work shows that the Hertz derivation can be understood as the introduction into electrodynamics of the principle which we have already formulated above. And at this stage the results of our work might be considered as the reinterpretation of the Hertz classical work on the basis of Galilean electrodynamics.

## 3.

Consider the motion of a dual charged particle with charges $q_1, g_1$ in the field of a particle with charges $q_2, g_2$. In the 'electric' limit one has the following equation:

$$m_1 \frac{d\mathbf{v}_1}{dt} = q_1 \mathbf{E}_2 + g_1 \mathbf{H}_2 + \mu_0 \mathbf{j}_{q_1} \times \mathbf{H}_{g_2} - \epsilon_0 \mathbf{j}_{g_1} \times \mathbf{E}_{q_2} \tag{3.1a}$$

In the rest system of particles 2 at time $t$ one finds:

$$\mathbf{H}'_{g_2} = \frac{g_2 \mathbf{r}}{4\pi\mu_0 r^3}, \qquad \mathbf{E}'_{q_2} = \frac{q_2 \mathbf{r}}{4\pi\epsilon_0 r^3}.$$

If particle 2 has the speed $\mathbf{v}_2$, transformation formula (1.9) provides us with fields measured in the laboratory

$$\mathbf{E}_2 = \frac{q_2 \mathbf{r}}{4\pi\epsilon_0 r^3} - \frac{g_2 \mathbf{v}_2 \times \mathbf{r}}{4\pi r^3}, \qquad \mathbf{H}_{g_2} = \mathbf{H}'_{g_2}$$

$$\mathbf{H}_2 = \frac{g_2 \mathbf{r}}{4\pi\mu_0 r^3} + \frac{q_2 \mathbf{v}_2 \times \mathbf{r}}{4\pi r^3}, \qquad \mathbf{E}_{q_2} = \mathbf{E}'_{q_2}$$

and there is the following Galilean invariant equation of motion:

$$m_1 \frac{d\mathbf{v}_1}{dt} = \left(\frac{q_1 q_2}{\epsilon_0} + \frac{g_1 g_2}{\mu_0}\right) \frac{\mathbf{r}}{4\pi r^3} + \frac{(q_1 g_2 - q_2 g_1)(\mathbf{v}_1 - \mathbf{v}_2) \times \mathbf{r}}{4\pi r^3} \tag{3.1b}$$

The equation of motion of particle 2 in the field produced by particle 1 can be readily obtained from (3.1b) (one must replace 1 by 2 and $\mathbf{r}$ by $-\mathbf{r}$)

† For a general account of theory of magnetic monopole, see the review article by Strazhev & Tomilchik (1973), where references to the original literature will be found.

$$m_2 \frac{d\mathbf{v}_2}{dt} = -\left(\frac{q_1 q_2}{\epsilon_0} + \frac{g_1 g_2}{\mu_0}\right) \frac{\mathbf{r}}{4\pi r^3} + \frac{(q_1 g_2 - q_2 g_1)(\mathbf{v}_2 - \mathbf{v}_1) \times \mathbf{r}}{4\pi r^3} \qquad (3.1c)$$

And, as can be seen, the condition

$$m_1 \frac{d\mathbf{v}_1}{dt} + m_2 \frac{d\mathbf{v}_2}{dt} = 0$$

that comes out to require the validity of Newton's third law is satisfied immediately.

An equation of the form (3.1b, c) was investigated by some authors (see, for example, Carter & Cohen, 1973) in the case of the theory of the magnetic monopole in non-relativistic approximation. It was stated there that one has no Galilean approach for the derivation of these equations, based on field theory. But, as we see, this approach does exist.

In the 'magnetic' limit one has

$$\nabla . {}^m\mathbf{j}_q = \nabla . {}^m\mathbf{j}_g = 0$$

so it is not possible to discuss in that case the motion of charged particles without supplementary assumptions. This question will be discussed in Section 4. But in this case one can state that the existence of the magnetic monopole is compatible with Galilean invariance. We note that the discussion of the motion of charged particles is the most appropriate in the 'electric' limit.

## 4.

We now consider the case when for electric sources and their fields use of the 'magnetic' limit is made and for magnetic sources and their fields use of the 'electric' limit is made. Introducing the definition:

$$\check{\mathbf{E}} = {}^e\mathbf{E}_g + {}^m\mathbf{E}_q$$
$$\check{\mathbf{H}} = {}^e\mathbf{H}_g + {}^m\mathbf{H}_q$$

one can formulate the Maxwell equations for field vectors $\mathbf{E}, \mathbf{H}$ with the help of equations for ${}^e\mathbf{E}_g, {}^e\mathbf{H}_g, {}^m\mathbf{E}_q, {}^m\mathbf{H}_q$ in corresponding limits. So one has

$$\nabla \times \check{\mathbf{H}} = {}^m\mathbf{j}_q \qquad \nabla . \check{\mathbf{E}} = {}^m\rho_q/\epsilon_0$$

$$\nabla \times \check{\mathbf{E}} = -\hat{\mu}_0 \frac{\partial \check{\mathbf{H}}}{\partial t} - {}^e\mathbf{j}_g, \qquad \nabla . \check{\mathbf{H}} = {}^e\rho_g/\mu_0 \qquad (4.1)$$

From (1.9) and (1.12) the Galilean transformations of fields $\check{\mathbf{E}}, \check{\mathbf{H}}$ can be derived:

$$\check{\mathbf{E}}' = \check{\mathbf{E}} + \mu_0 \mathbf{v} \times \check{\mathbf{H}}$$
$$\check{\mathbf{H}}' = \check{\mathbf{H}} \qquad (4.2)$$

The sources ${}^m\mathbf{j}_q, {}^m\rho_q, {}^e\mathbf{j}_g, {}^e\rho_g$ are transformed in accordance with (1.9) and (1.12). The Maxwell equations (4.1) are invariants under transformations

(1.10), (4.2) and Galilean transformations of sources. The Galilean invariant expression for force has in this mixture limit the following form:

$$\check{\mathbf{F}} = \int d^3r\{{}^e\rho_g\check{\mathbf{H}}(\mathbf{r}) + \mu_0{}^m\mathbf{j}_q \times \check{\mathbf{H}}(\mathbf{r})\} \tag{4.3}$$

For the magnetic sources, but not for the electric sources, the continuity equation is held in this mixture limit. We must suppose that the vector density ${}^m\mathbf{j}_q$ is a source for magnetic fields, a source which has nothing to do with electric charges (cf. Bacry & Kubar-Andre, 1973). In this case an electric charge in motion cannot produce a magnetic field; we can consider, simultaneously, two kinds of sources: ${}^m\rho_q$, ${}^m\mathbf{j}_q$ and ${}^e\rho_q$, ${}^e\mathbf{j}_q$.

The Galilean invariant expression of force has in this case the form:

$$\check{\mathbf{F}} = \int d^3r\{{}^e\rho_g\check{\mathbf{H}}(\mathbf{r}) + \mu_0{}^m\mathbf{j}_q \times \check{\mathbf{H}}(\mathbf{r}) + \mu_0{}^e\mathbf{j}_q \times \check{\mathbf{H}}(\mathbf{r}) + {}^e\rho_q . \check{\mathbf{E}}(\mathbf{r})\} \tag{4.4}$$

We note here that in accordance with the above-mentioned assumption the term ${}^e\rho_g . {}^e\mathbf{H}_q$ in (4.4) must be dropped. Without this term one cannot include the term ${}^e\mathbf{j}_g \times {}^e\mathbf{E}_q$ in (4.4) because the term is not Galilean invariant when alone. This circumstance is very important in the following work. It would be logical, of course, to formulate the Maxwell equations and Galilean transformations for the fields $\check{\mathbf{H}}, '\check{\mathbf{E}}$, where $'\check{\mathbf{E}} = \check{\mathbf{E}} + {}^e\check{\mathbf{E}}_q$, and introduce in equation

$$\nabla . \check{\mathbf{E}} = {}^m\rho_q | \epsilon_0$$

two types of sources of an electric field:

$$\nabla . '\check{\mathbf{E}} = ({}^m\rho_q + {}^e\rho_q)/\epsilon_0$$

Then in the expression for force one should take, instead of the term ${}^e\rho_q\check{\mathbf{E}}$, the term ${}^e\rho_q{}'\check{\mathbf{E}}$. But this modification has no principal value for the following discussion and for simplicity our attention will be concentrated on equations (4.1) and (4.4). If one wants to consider the mutual interaction of dual charged particles then in (4.4) the term $\mu_0{}^m\mathbf{j}_q \times \check{\mathbf{H}}$ ought to be omitted. The equations (4.1) and (4.4) are the basis of the work of Bacry & Kubar-Andre (1973). For the case of interaction of two dual charged particles we have (in line of reasoning with Section 3):

$$m_1 \frac{d\mathbf{v}_1}{dt} = \left(\frac{q_1q_2}{\epsilon_0} + \frac{g_1g_2}{\mu_0}\right)\frac{\mathbf{r}}{4\pi r^3} + q_1g_2\frac{(\mathbf{v}_1 - \mathbf{v}_2) \times \mathbf{r}}{4\pi r^3}$$

$$m_2 \frac{d\mathbf{v}_2}{dt} = -\left(\frac{q_1q_2}{\epsilon_0} + \frac{g_1g_2}{\mu_0}\right)\frac{\mathbf{r}}{4\pi r^3} - g_1q_2\frac{(\mathbf{v}_2 - \mathbf{v}_1) \times \mathbf{r}}{4\pi r^3} \tag{4.5}$$

and the validity of Newton's third law is required:

$$g_1q_2 + g_2q_1 = 0 \tag{4.6}$$

So from the condition (4.6), as was shown by Bacry & Kubar-Andre (1973), it follows that in Galilean invariant theory there is no place for dual charged particles. This conclusion can also be stated in the case where electric and magnetic charges exist separately. The reason for these conclusions is made very clear in our approach. For the case of dual charged particles this conclusion is caused by the application of two mutually excluding physical conditions (1.6) and (1.11) to the same particle. Of course, one can formulate Galilean theory on the basis of uniting two Galilean limits for every type of source. But in this case, as was shown by Le Bellac & Levy-Lebbond (1973), the fulfilment of Newton's third law is not required. This means that this law is consistent with Galilean invariance but is not obligatory.

The approach of Bacry & Kubar-Andre (1973) can, in principle, be used in the case of the separate existence of electric and magnetic charges. But from the general point of view this approach should be rejected. As was noted by Le Bellac & Levy-Leblond (1973), the use of the limiting procedure $c \to \infty$ in the Maxwell equations, formulated in the system of units including the velocity of light in their definition (CGSE or CGSM), is very ambiguous.

The results of the work of Bacry and Kubar-Andre arise from the use of system of units CGSM which is not appropriate for deriving the Galilean limits of electrodynamics of dual charged particles. In the last section, Section IV of their work an analysis of the theory is given which comes close to the initial position of our work. But the principal difference lies in the fact that in our approach the restrictions (1.6) and (1.11) on the fields are formulated for every type of field $\mathbf{E}_q, \mathbf{E}_g, \mathbf{H}_q, \mathbf{H}_g$. If we do not take into account the difference† between the fields $\mathbf{E}_q, \mathbf{H}_q$ and $\mathbf{E}_g, \mathbf{H}_g$ then the conditions (1.6) and (1.11) for the fields are not self-consistent, as was in fact stated by the above-mentioned authors.

## 5.

It may seem somewhat strange that the constants $\epsilon_0, \mu_0$ appear in the electric and magnetic limits simultaneously. It is known that in the MKSA type system these constants are related through the formula $\epsilon_0\mu_0 = 1/c^2$. And if we can measure the quantities $\epsilon_0$ and $\mu_0$ simultaneously, it means that in principle we can measure the velocity of light. But this possibility is contrary to our definition of Galilean theory.

But this time it can be clearly seen that the wave equations there have the form ($\rho = 0, \mathbf{j} = 0$)

$$\nabla^2 \mathscr{F} = 0 \tag{5.1}$$

where $\mathscr{F} \equiv (\mathbf{E}, \mathbf{H})$ and $\mathbf{E}, \mathbf{H}$ are the vectors of electric or magnetic fields in the corresponding limits. In equation (5.1) the term $\epsilon_0\mu_0(\partial^2\mathbf{F}/\partial t^2)$ is absent and the factor $\epsilon_0\mu_0$ cannot be compared with the $1/c^2$ factor. This fact can be explained in the following way. The relation between $\epsilon_0$ and $\mu_0$ in the electro-

† We have already seen that consideration of fields $\mathbf{E}_q, \mathbf{E}_g$ and $\mathbf{H}_q, \mathbf{H}_g$ on equal terms is a feature of Maxwell electrodynamics but not Galilean electromagnetism.

dynamics of Maxwell can be stated only on the basis of knowing the laws of Kulon, Bio-Savar and the equation of continuity for the sources. But none of the Galilean limits of electrodynamics has these laws and satisfies the equation of continuity simultaneously.

The introduction of interaction between electric and magnetic sources does not change the whole situation. The reason is that we introduce a new unit of dimension: the dimension of magnetic charge ($G$). From this point of view it is more logical, of course, to express the Maxwell equations in the system of five units: Cohn system (Cohn, 1900), Sommerfeld (1967) or MKSQG system of units. The Maxwell equations have the following form in the Cohn system of units.

$$\Gamma \nabla \times \mathbf{H} = \epsilon_0' \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j}_q, \qquad \nabla . \mathbf{E} = \rho_q / \epsilon_0'$$

$$\Gamma \nabla \times \mathbf{E} = -\mu_0' \frac{\partial \mathbf{H}}{\partial t} - \mathbf{j}_g, \qquad \nabla . \mathbf{H} = \rho_g /= \mu_0' \tag{5.2}$$

Here $\Gamma$ is a new absolute constant with dimension $[\Gamma] = QG^{-1}\, m^1\, \text{sek}^{-1}$ and between $\Gamma$, $\epsilon_0'$, $\mu_0'$ and $c$ the following relation is stated

$$\Gamma \epsilon_0' \mu_0' = 1/c^2$$

If one puts $\Gamma$ equal to 1 then one can make the transition to the MKSQ system. But this includes the idea that the static interaction between magnetic charges can be described as the interaction of two permanent magnets (on the basis of electric sources) under appropriate conditions. But this possibility cannot be realised in both Galilean limits because, in this case, it means the physical equivalence of magnetic fields $\mathbf{H}_q$ and $\mathbf{H}_g$. From this consideration it becomes clear that the simultaneous presence of $\epsilon_0$ and $\mu_0$ in the equations of both Galilean limits is innocuous and does not indicate the possibility of the definition of velocity of light.

Another way of describing the situation involves the idea that one can measure the fields $\mathbf{E}_q$, $\mathbf{E}_g$ and $\mathbf{H}_q$, $\mathbf{H}_g$, both limits in different units. Taking into consideration the last remark, one can easily see the method of obtaining the system of equations (1.7) and (1.8) with the help of the limiting procedure $c \to \infty$ in the Maxwell equations. In this case, for example, the 'electric' limit can be obtained by taking equations (1.4) in the CGSE system and equations (1.5) in the CGSM system. In general the use of the Cohn system of units for the investigation of Galilean limits of electrodynamics with magnetic charges is, in principle, preferable.

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